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Dynamical systems and quantum bicrossproduct algebras

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Abstract
We present a unified study of some aspects of quantum bicrossproduct algebras of inhomogeneous Lie algebras, such as Poincaré, Galilei and Euclidean in N dimensions. The action associated with the bicrossproduct structure allows us to obtain a nonlinear action over a new group linked to the translations. This new nonlinear action associates a dynamical system with each generator which is the object of our study.

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1. Introduction

In a series of papers [1–5] we have dealt with the problem of the construction of induced representations of quantum inhomogeneous algebras. In particular, in [4] we have focussed our attention on quantum Hopf algebras having the structure of bicrossproduct $\mathcal{H} = U(\mathcal{K}) \triangleright\!\!\blacktriangleleft U_z(\mathcal{L})$, with $U(\mathcal{K})$ a cocommutative Hopf algebra, $U_z(\mathcal{L})$ a commutative but noncocommutative Hopf algebra and $\mathcal{K}$ and $\mathcal{L}$ Lie algebras [6]. Remember that this bicrossproduct structure is the deformed counterpart of the semidirect product of Lie groups $(H = L \odot K)$. In this paper, we want to utilize some of the techniques developed in the above-mentioned papers to obtain relevant information about some aspects related to the bicrossproduct algebras which are the object of our study.

We shall reinterpret the above bicrossproduct structure as $H = U(\mathcal{K}) \triangleright\!\!\blacktriangleleft \mathrm{Fun}(L_z)$, because the commutativity of $U\mathcal{L}$ allows us to identify it with the algebra of functions over a certain group L_z. The bicrossproduct structure determines an action of $U(\mathcal{K})$ on $\mathrm{Fun}(L_z)$ which at the level of groups originates a nonlinear action of the group K on L_z. At the infinitesimal level this last action is described by vector fields associated with the generators of $\mathcal{K}$. These vector fields give rise to one-parameter flows, some of them being linear ('nondeformed') and others nonlinear ('deformed'). In other words, we can study some dynamical systems associated with this action.

As is well known in the nondeformed case, the homogeneous space $X = H/K$ is diffeomorphic to $\mathbb{R}^N$ which is associated with L, N being the dimension of L. In the quantum case the homogeneous space is now identified with Fun(L_z). However, we can study the nonlinear action of K on L_z. Note that in the limit $z \to 0$ we recover the linear action of K on X.

We will consider the family of inhomogeneous algebras related by graded contractions to the compact algebra $so(N+1)$ [7, 8]. They are called inhomogeneous Cayley–Klein algebras. Among the elements of this family we find the Poincaré and the Galilei algebras in $(N-1, 1)$ dimensions and the Euclidean algebra in N dimensions. The bicrossproduct structure that shares these quantum algebras [9] allows us to present a unified study of the properties mentioned above.

The organization of the paper is as follows. Section 2 presents a brief review of the inhomogeneous Cayley–Klein algebras, their quantum deformations and their bicrossproduct structure. The next section, the most important part of the work, is devoted to computing the flow associated with the action, the invariant under this action that coincides with the Casimir and the dynamical systems associated with it. In section 4 we present, as an example, the case of $N = 3$ to illustrate the ideas introduced in the previous section. Some graphics show the different leaves associated with the action and their foliations. We complete with some conclusions and remarks.

2. Quantum Cayley–Klein algebras $U_z\big(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N)\big)$

The family of Cayley–Klein pseudo-orthogonal algebras is a set of $(N+1)N/2$-dimensional real Lie algebras characterized by N real parameters $(\omega_1, \omega_2, \ldots, \omega_N)$ and denoted by $\mathfrak{so}_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ [7, 8]. In an appropriate basis $(J_{ij})_{0\leqslant i<j\leqslant N}$ the nonvanishing commutators are

$$[J_{ij}, J_{ik}] = \omega_{ij} J_{jk} \qquad [J_{ij}, J_{jk}] = -J_{ik} \qquad [J_{ik}, J_{jk}] = \omega_{jk} J_{ij}$$

with the subindices verifying $0 < i < j < k < N$ and $\omega_{ij} = \Pi_{s=i+1}^{j}\omega_s$. The generators can be rescaled in such a way that the parameters ω_i only take the values $1, 0$ and -1. When all the ω_i are different from zero the algebra $\mathfrak{so}_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ is isomorphic to some of the pseudo-orthogonal algebras $\mathfrak{so}(p, q)$ with $p + q = N + 1$ and $p \geqslant q > 0$. If some of the coefficients ω_i vanish the corresponding algebra is inhomogeneous and can be obtained from $\mathfrak{so}(p, q)$ by means of a sequence of contractions. In the particular case of $\omega_1 = 0$, the algebras $\mathfrak{so}_{0,\omega_2,\ldots,\omega_N}(N+1)$ can be realized as algebras of groups of affine transformations on $\mathbb{R}^N$ [7]. In this case, the generators J_{0i} are denoted by P_i, stressing, in this way, their role as generators of translations. The remaining generators J_{ij} originate compact and 'noncompact' rotations. These inhomogeneous algebras, henceforth denoted by $\mathfrak{iso}_{\omega_2,\ldots,\omega_N}(N)$, are characterized by the following nonvanishing commutators:

$$\begin{array}{lll} [J_{ij}, J_{ik}] = \omega_{ij} J_{jk} & [J_{ij}, J_{jk}] = -J_{ik} & [J_{ik}, J_{jk}] = \omega_{jk} J_{ij} \\ [J_{ij}, P_i] = P_j & [J_{ij}, P_i] = -\omega_{ij} P_i & 1 \leqslant i < j < k \leqslant N. \end{array}$$

In [10, 11], simultaneous standard deformations (i.e. their associated classical r-matrices are quasi-triangular [12]) for all the enveloping algebras $U(\mathfrak{so}_{\omega_1,\omega_2}(3))$ and $U(\mathfrak{so}_{\omega_1,\omega_2,\omega_3}(4))$, respectively, were introduced. In [13], the case of $U(\mathfrak{iso}_{\omega_2,\omega_3,\omega_4}(4))$ was considered, and the general case $U(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ (all of them standard deformations) was considered in [14].

It was proved in [9] that the standard quantum Hopf algebras $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ have a structure of a bicrossproduct. Using a basis adapted to the bicrossproduct structure we can describe together all these quantum algebras $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$. In order to avoid repetitions we use the following convention: the variation rank of i, j, k is $1, \ldots, N-1$ and the index N is treated separately. Besides, when two indices i, j appear in a generator it is assumed that $i < j$. The commutation relations are

$$
\begin{array}{lll}
[P_i, P_j] = 0 & [P_i, P_N] = 0 & \\
[J_{ij}, J_{ik}] = \omega_{ij} J_{jk} & [J_{ij}, J_{jk}] = -J_{ik} & [J_{ik}, J_{jk}] = \omega_{jk} J_{ij} \\
[J_{ij}, J_{iN}] = \omega_{ij} J_{jN} & [J_{ij}, J_{jN}] = -J_{iN} & [J_{ik}, J_{jN}] = \omega_{jN} J_{ij} \\
[J_{ij}, P_k] = \delta_{ik} P_k - \delta_{jk}\omega_{ij} P_i & [J_{ij}, P_N] = 0 &
\end{array}
\tag{2.1}
$$

$$
[J_{iN}, P_j] = \delta_{ij} \left(\frac{1 - \mathrm{e}^{-2zP_N}}{2z} - \frac{z}{2} \sum_{s=1}^{N-1} \omega_{sN} P_s^2 \right) + z\omega_{iN} P_i P_j \qquad [J_{iN}, P_N] = -\omega_{iN} P_i
$$

and the coproduct is given by

$$
\begin{aligned}
&\Delta(P_i) = P_i \otimes 1 + \mathrm{e}^{-zP_N} \otimes P_i \qquad \Delta(P_N) = P_N \otimes 1 + 1 \otimes P_N \\
&\Delta(J_{ij}) = J_{ij} \otimes 1 + 1 \otimes J_{ij} \\
&\Delta(J_{iN}) = J_{iN} \otimes 1 + \mathrm{e}^{-zP_N} \otimes J_{ij} + z \sum_{s=1}^{i-1} \omega_{iN} P_s \otimes J_{si} - z \sum_{s=i+1}^{N-1} \omega_{sN} P_s \otimes J_{is}.
\end{aligned}
$$

The bicrossproduct structure $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N)) = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$, with $\mathcal{K} = U(\mathfrak{so}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ and $\mathcal{L}$ the commutative Hopf subalgebra generated by $P_1, P_2, \ldots, P_N$, is described by the right action of $\mathcal{K}$ over $\mathcal{L}$

$$
P_i \triangleleft J_{jk} = [P_i, J_{jk}] \qquad j < k \quad i, j, k = 1, 2, \ldots, N
$$

with the commutators given by (2.1), and the left coaction of $\mathcal{L}$ over $\mathcal{K}$, whose expression over the generators of $\mathcal{K}$ is

$$
\begin{aligned}
&J_{ij} \blacktriangleleft\!\!= 1 \otimes J_{ij} \\
&J_{iN} \blacktriangleleft\!\!= \mathrm{e}^{-zP_N} \otimes J_{iN} + z \sum_{s=1}^{i-1} \omega_{iN} P_s \otimes J_{si} - z \sum_{s=i+1}^{N-1} \omega_{sN} P_s \otimes J_{is}.
\end{aligned}
$$

3. One-parameter flows

In [9], the algebra $U_z(T_N)$ was considered as a noncommutative deformation of the Lie algebra of the group of translations of $\mathbb{R}^N$. However, here we can utilize the commutativity of $U_z(T_N)$, interpreting it as the algebra of functions over a group, in such a way that we have the following bicrossproduct decomposition:

$$
U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N)) = U(\mathfrak{so}_{\omega_2,\omega_3,\ldots,\omega_N}(N)) \triangleright\!\!\blacktriangleleft F(T_{z,N})
$$

where $T_{z,N}$ is the space $\mathbb{R}^N$ equipped with the composition law

$$
\begin{aligned}
&(\alpha_1', \alpha_2', \ldots, \alpha_{N-1}', \alpha_N')(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N) \\
&\qquad = \left(\alpha_1' + \mathrm{e}^{-z\alpha_N'}\alpha_1, \alpha_2' + \mathrm{e}^{-z\alpha_N'}\alpha_2, \ldots, \alpha_{N-1}' + \mathrm{e}^{-z\alpha_N'}\alpha_{N-1}, \alpha_N' + \alpha_N\right)
\end{aligned}
$$

that equips it with a structure of N-dimensional Lie group. The group $T_{z,N}$ has the structure of a semidirect product of the additive groups $\mathbb{R}^{N-1}$ and $\mathbb{R}$

$$
T_{z,N} \equiv \mathbb{R}^{N-1} \rtimes \mathbb{R} \qquad (a', b')(a, b) = (a' + a \triangleleft (-b), b' + b)
$$

where the right action of $\mathbb{R}$ over $\mathbb{R}^{N-1}$ is given by means of the usual product over each component,

$$a \triangleleft b = \mathrm{e}^{zb} a \qquad a \in \mathbb{R}^{N-1} \quad b \in \mathbb{R}.$$

The generators P_i of $U_z(T_N)$ give in this context a global chart over $T_{z,N}$,

$$P_i(\alpha) = \alpha_i \qquad \alpha \in T_{z,N}.$$

The structure of $U(\mathfrak{so}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$-module algebra of $F(T_{z,N})$ implies that an action of the group $SO_{\omega_2,\omega_3,\ldots,\omega_N}(N)$ on $T_{z,N}$ is defined. At the infinitesimal level this action is described by the vector fields

$$\hat{J}_{ij} = -P_j \frac{\partial}{\partial P_i} + \omega_{ij} P_i \frac{\partial}{\partial P_j}$$

$$\begin{aligned}\hat{J}_{iN} &= \sum_{j=1}^{N-1} - \left[\delta_{ij}\left(\frac{1-\mathrm{e}^{-2zP_N}}{2z} - \frac{z}{2}\sum_{s=1}^{N-1}\omega_{sN}P_s^2\right) + z\omega_{iN}P_iP_j\right]\frac{\partial}{\partial P_j} + \omega_{iN}P_i\frac{\partial}{\partial P_N}\\ &= \sum_{j\neq i,N} - z\omega_{iN}P_iP_j\frac{\partial}{\partial P_j} - \left[\frac{1-\mathrm{e}^{-2zP_N}}{2z}\right.\\ &\qquad \left. - \frac{z}{2}\sum_{s=1}^{N-1}\omega_{sN}P_s^2 + z\omega_{iN}P_i^2\right]\frac{\partial}{\partial P_i} + \omega_{iN}P_i\frac{\partial}{\partial P_N}.\end{aligned} \tag{3.1}$$

Since only the generators $J_{1N}, J_{2N}, \ldots, J_{N-1N}$ have deformed action the integration of the equations of the fields $\hat{J}_{ij}$ gives the well-known linear flows

$$\Phi_{ij}^t(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots,\alpha_N) = (\alpha_1,\ldots,\alpha_{i-1},\alpha_i',\alpha_{i+1},\ldots,\alpha_{j-1},\alpha_j',\alpha_{j+1},\ldots,\alpha_N) \tag{3.2}$$

with

$$\alpha_i' = C_{\omega_{ij}}(t)\alpha_i - S_{\omega_{ij}}(t)\alpha_j \qquad \alpha_j' = \omega_{ij}S_{\omega_{ij}}(t)\alpha_i + C_{\omega_{ij}}(t)\alpha_j$$

where

$$C_\omega(t) = \frac{\mathrm{e}^{\sqrt{-\omega}t} + \mathrm{e}^{-\sqrt{-\omega}t}}{2} \qquad S_\omega(t) = \frac{\mathrm{e}^{\sqrt{-\omega}t} - \mathrm{e}^{-\sqrt{-\omega}t}}{2\sqrt{-\omega}}.$$

So, we have simple compact or noncompact rotations in the ij plane.

The computation of the flows associated with the 'deformed' fields $\hat{J}_{iN}$ requires a more careful analysis. Let us start by obtaining their invariants. Supposing that the differential form

$$\eta = \sum_{s=1}^{N} \mu_s \,\mathrm{d}P_s \tag{3.3}$$

verifies $\hat{J}_{iN} \rfloor \eta = 0$, the following equation is obtained:

$$\sum_{j\neq i,N} z\omega_{iN}P_iP_j\mu_j + \left[\frac{1-\mathrm{e}^{-2zP_N}}{2z} - \frac{z}{2}\sum_{s=1}^{N-1}\omega_{sN}P_s^2 + z\omega_{iN}P_i^2\right]\mu_i - \omega_{iN}P_i\mu_N = 0. \tag{3.4}$$

Using this expression $(N-1)$ invariant functions are obtained as follows. For the first invariant we choose

$$\mu_s = \omega_{sN}P_s\tau \qquad s = 1, 2, \ldots, N-1$$

with τ a function to be evaluated. Hence, equation (3.4) reduces to

$$\omega_{iN} P_i \left[\frac{1 - \mathrm{e}^{-2zP_N}}{2z} + \frac{z}{2} \sum_{s=1}^{N-1} \omega_{sN} P_s^2 \right] \tau - \omega_{iN} P_i \mu_N = 0. \tag{3.5}$$

From equation (3.5) we find the value of μ_N obtaining the differential form

$$\eta = \tau \left[\sum_{j=1}^{N-1} \omega_{jN} P_j \, \mathrm{d}P_j + \left(\frac{1 - \mathrm{e}^{-2zP_N}}{2z} + \frac{z}{2} \sum_{s=1}^{N-1} \omega_{sN} P_s^2 \right) \mathrm{d}P_N \right]$$

where τ plays the role of an integration factor. Solving the case $N = 2$ we get $\tau = 2\,\mathrm{e}^{zP_N}$, which is proved to be valid for every N. The integration of the equations

$$\frac{\partial h}{\partial P_s} = \mu_s \qquad 1 \leqslant s \leqslant N$$

gives $\eta = \mathrm{d}h$. By an appropriate choice of the integration constant, in order to have good behaviour in the limit $z \to 0$, we obtain

$$h_{\omega,z} = \sum_{j=1}^{N-1} \omega_{jN} P_j^2 \, \mathrm{e}^{zP_N} + \frac{\cosh(zP_N) - 1}{\frac{z^2}{2}}. \tag{3.6}$$

This function is, in fact, invariant under the action of all the generators J_{ij}. Indeed, it belongs to the centre of the algebra $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ and is the Casimir C_z given in [9], but now it appears in a natural way.

To obtain the other $N-2$ invariants we start fixing $k \in \{1, 2, \ldots, N-1\} - \{i\}$ and taking $\mu_j = 0$ if $j \neq k, N$, we get from expression (3.3) the differential form $\eta_k = \mu_k \, \mathrm{d}P_k + \mu_N \, \mathrm{d}P_N$. Condition (3.4) applied to η_k establishes a relationship between μ_k and μ_N that allows us to write

$$\eta_k = \mu_k \, \mathrm{d}P_k + \mu_k z P_k \, \mathrm{d}P_N.$$

Choosing $\mu_k = \mathrm{e}^{zP_N}$ the differential form is exact, that is, $\eta_k = \mathrm{d}h_{\omega,z}^{iN,k}$, with

$$h_{\omega,z}^{iN,k} = P_k \, \mathrm{e}^{zP_N} \qquad k \in \{1, 2, \ldots, N-1\} - \{i\}. \tag{3.7}$$

To obtain the integral curves of $\hat{J}_{iN}$ it is necessary to solve the system of N differential equations

$$\begin{aligned} \dot{\alpha}_j &= -z\omega_{iN}\alpha_i\alpha_j \qquad j \neq i, N \\ \dot{\alpha}_i &= -\frac{1 - \mathrm{e}^{-2z\alpha_n}}{2z} + \frac{z}{2} \sum_{s=1}^{N-1} \omega_{sN}\alpha_s^2 - z\omega_{iN}\alpha_i^2 \\ \dot{\alpha}_N &= \omega_{iN}\alpha_i. \end{aligned}$$

The invariants $h_{\omega,z}^{iN,k}$ (3.7) allow us to remove $N-2$ degrees of freedom, from $h_{\omega,z}^{iN,k}(\alpha) = \alpha_k \, \mathrm{e}^{z\alpha_N} = \beta_k$ we obtain $\alpha_k = \beta_k \, \mathrm{e}^{-z\alpha_N}$, restricting the study of the N-dimensional system to the following family of two-dimensional systems depending on the N parameters β_k, ω and z

$$\begin{aligned} \dot{\alpha}_i &= -\left[\frac{1 - \mathrm{e}^{-2z\alpha_N}}{2z} - \frac{z}{2} \left(\sum_{s \neq i,N} \omega_{sN}\beta_s^2 \right) \mathrm{e}^{-2z\alpha_N} + \frac{z}{2}\omega_{iN}\alpha_i^2 \right] \\ \dot{\alpha}_N &= \omega_{iN}\alpha_i. \end{aligned} \tag{3.8}$$

Because of the way in which the parameters β_k appear grouped, the set of systems (3.8) only depends on three parameters z, ω_{iN} and $\rho = \sum_{s\neq i,N}\omega_{sN}\beta_s^2$. The function $h_{\omega,z}$ (3.6) gives the following invariant for the system (3.8):

$$\omega_{iN}\alpha_i^2\,\mathrm{e}^{z\alpha_N} + \rho\,\mathrm{e}^{-z\alpha_N} + \frac{\cosh(z\alpha_N) - 1}{\frac{z^2}{2}}. \tag{3.9}$$

The description of the systems when $z = 0$ is trivial, since it reduces to the study of linear systems analogous to those of the fields $\hat{J}_{ij}$. If z does not vanish the equations may be rescaled considering

$$x(t) = z\alpha_i(t) \qquad y(t) = z\alpha_N(t)$$

and setting $a = \omega_{iN}$, $b - 1 = z^2\rho = z^2\sum_{s\neq i,N}\omega_{sN}\beta_s^2$ the system becomes

$$\dot{x} = -\tfrac{1}{2}ax^2 - \tfrac{1}{2} + \tfrac{1}{2}b\,\mathrm{e}^{-2y} \qquad \dot{y} = ax. \tag{3.10}$$

In this form the limit $z \to 0$ cannot be studied, but as an advantage it depends on only two parameters. The possibility of reabsorption of the parameter z follows from the fact that all the Hopf algebras $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ are isomorphic (for fixed values of the parameters ω_s) whenever z is nonzero. The function (3.9) gives rise to the following invariant of (3.10):

$$h_{a,b} = ax^2\,\mathrm{e}^y + \mathrm{e}^y + b\,\mathrm{e}^{-y}. \tag{3.11}$$

The research of the fixed points of the system reveals that:

- if $b \leqslant 0$ the system does not have equilibrium points;
- if $b > 0$ there are three possibilities:
  - if $a < 0$ then there is only one fixed point $\left(0, \frac{1}{2}\ln b\right)$ of hyperbolic character,
  - if $a = 0$ then all the points such as $\left(x, \frac{1}{2}\ln b\right)$ are fixed points,
  - if $a > 0$ there is only one equilibrium point $\left(0, \frac{1}{2}\ln b\right)$ of elliptic character.

Let us analyse in detail the case $a > 0$ and $b > 0$. Here, invariant (3.11) has a global minimum value $2\sqrt{b}$ at $\left(0, \frac{1}{2}\ln b\right)$ and it is easy to check that $h_{a,b}$ takes arbitrarily high values over points going to infinity in any direction. Since the orbits of the system (3.10) are the level curves of $h_{a,b}$ all the orbits are bounded. Note that the point of equilibrium disappears in the limit $b \to 0$. Let us consider the integral curve γ_r passing through the point $(0, r)$, with $r > \frac{1}{2}\ln b$, at the initial time. For small values of $t > 0$ the invariant allows us to obtain x in terms of y

$$ax = -\sqrt{a\,\mathrm{e}^{-y}(\mathrm{e}^r + b\,\mathrm{e}^{-r} - \mathrm{e}^y - b\,\mathrm{e}^{-y})} \tag{3.12}$$

in such a way that substituting this in the second equation of the system (3.10) is enough to do a quadrature. The final result gives the following expression for the curve γ_r:

$$\gamma_r(t) = \left(\frac{-(\mathrm{e}^r - b\,\mathrm{e}^{-r})S_a(t)}{(\mathrm{e}^r + b\,\mathrm{e}^{-r}) + (\mathrm{e}^r - b\,\mathrm{e}^{-r})C_a(t)}, \ln\frac{1}{2}\left[(\mathrm{e}^r + b\,\mathrm{e}^{-r}) + (\mathrm{e}^r - b\,\mathrm{e}^{-r})C_a(t)\right]\right). \tag{3.13}$$

From (3.13) the flow associated with the system (3.10) is obtained provided $a > 0$ and $b > 0$

$$\begin{aligned}\Phi^t_{a,b}(x,y) = \Bigg(&\frac{(ax^2\,\mathrm{e}^y - \mathrm{e}^y + b\,\mathrm{e}^{-y})S_a(t) + (2x\,\mathrm{e}^y)C_a(t)}{(ax^2\,\mathrm{e}^y + \mathrm{e}^y + b\,\mathrm{e}^{-y}) + (-ax^2\,\mathrm{e}^y + \mathrm{e}^y - b\,\mathrm{e}^{-y})C_a(t) + (2ax\,\mathrm{e}^y)S_a(t)},\\ &\ln\frac{1}{2}\left[(ax^2\,\mathrm{e}^y + \mathrm{e}^y + b\,\mathrm{e}^{-y}) + (-ax^2\,\mathrm{e}^y + \mathrm{e}^y - b\,\mathrm{e}^{-y})C_a(t) + (2ax\,\mathrm{e}^y)S_a(t)\right]\Bigg).\end{aligned} \tag{3.14}$$

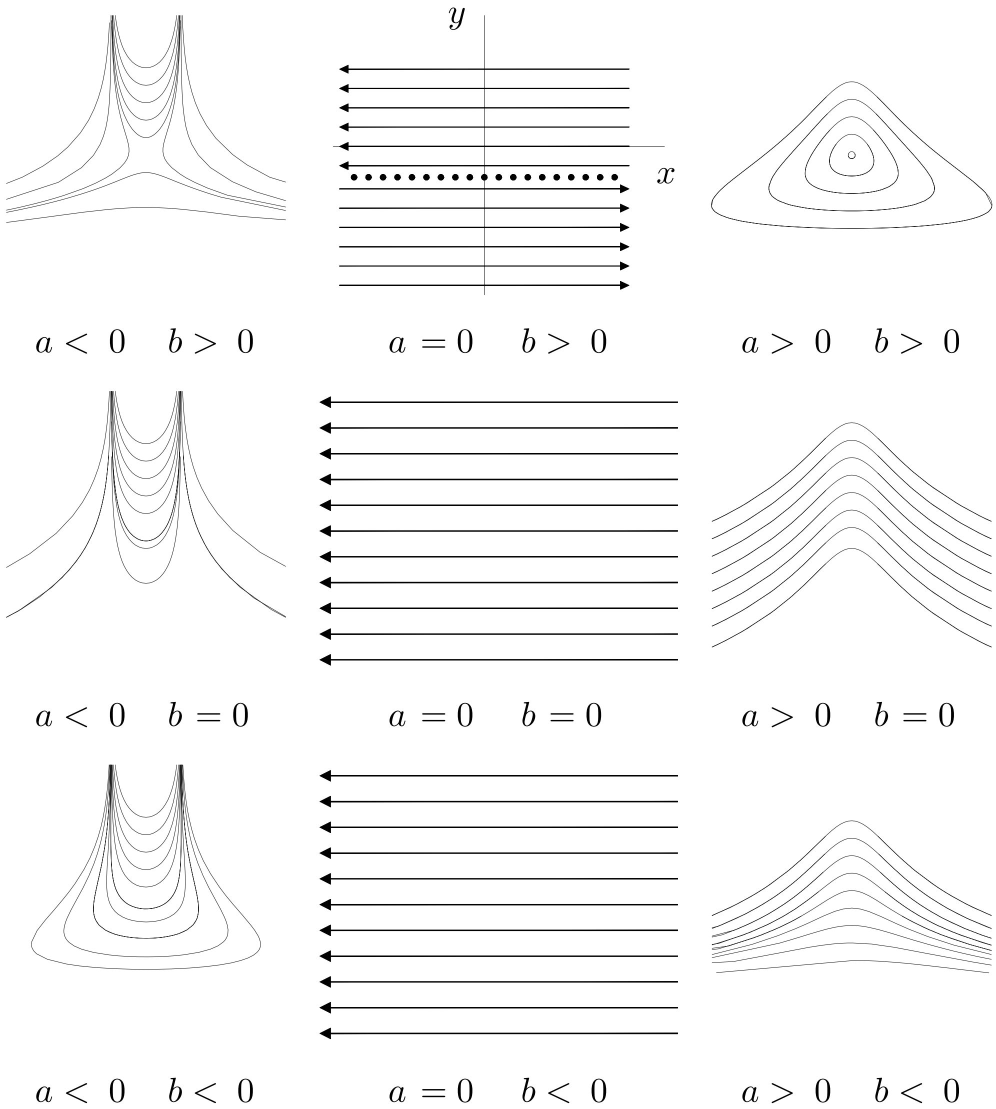

Figure 1. Qualitative aspect of the orbits associated with the flow $\Phi_{a,b}$.

It is easy to prove that (3.14) is also valid for the remaining values of a and b. Note that, if the parameters a and b are positive then the flow is defined globally, but this does not happen, in general, for any other value of the parameters.

Figure 1 shows the qualitative form of the integral curves for different values of a and b. Besides the reflection symmetry with respect to the vertical axis, it is worth noting the pair of straight lines of equations $x = 1/\sqrt{-a}$ and $x = -1/\sqrt{-a}$, which are present in the cases $a < 0$ and $b = 0$, although they are not displayed in the figure, separating the lower bounded orbits from the unbounded ones. Also note that when $a > 0$ and $b = 0$ all the curves are obtained translating vertically the graphic of the function $y = -\ln(1 + ax^2)$.

The preceding study allows us to write the flow $\Phi^t_{iN} : T_{z,N} \to T_{z,N}$ of the vector field $\hat{J}_{iN}$ (3.1). For its description it is convenient to introduce the functions $F^{\omega,z}_{iN} : T_{z,N} \times \mathbb{R} \to \mathbb{R}$, defined by

$$F_{iN}^{\omega,z}(\alpha,t) = \left[\cosh(z\alpha_N) + \frac{z^2}{2}\sum_{s=1}^{N-1}\omega_{sN}\alpha_s^2\,\mathrm{e}^{z\alpha_N}\right] + \left[\sinh(z\alpha_N) - \frac{z^2}{2}\sum_{s=1}^{N-1}\omega_{sN}\alpha_s^2\,\mathrm{e}^{z\alpha_N}\right]C_{\omega_{iN}}(t) + \left[z\omega_{iN}\alpha_i\,\mathrm{e}^{z\alpha_N}\right]S_{\omega_{iN}}(t).$$

Note that the first term can be written in terms of the invariant $h_{\omega,z}$ as

$$\cosh(z\alpha_N) + \frac{z^2}{2}\sum_{s=1}^{N-1}\omega_{sN}\alpha_s^2\,\mathrm{e}^{z\alpha_N} = 1 + \frac{z^2}{2}h_{\omega,z}(\alpha).$$

Writing the flow action as

$$\Phi_{iN}^t(\alpha) = \alpha' \tag{3.15}$$

we get

$$\alpha_i' = \frac{-\left[\sinh(z\alpha_N) - \frac{z^2}{2}\sum_{s=1}^{N-1}\omega_{sN}\alpha_s^2\,\mathrm{e}^{z\alpha_N}\right]S_{\omega_{iN}}(t) + \left[z\alpha_i\,\mathrm{e}^{z\alpha_N}\right]C_{\omega_{iN}}(t)}{zF_{iN}^{\omega,z}(\alpha,t)}$$

$$\alpha_N' = \frac{1}{z}\ln F_{iN}^{\omega,z}(\alpha,t)$$

$$\alpha_j' = \frac{\alpha_j\,\mathrm{e}^{z\alpha_N}}{F_{iN}^{\omega,z}(\alpha,t)} \qquad j \neq i, N.$$

The limit $z \to 0$ can be obtained after considering the first order in z of the function $F_{iN}^{\omega,z}$:

$$F_{iN}^{\omega,z}(\alpha,t) = 1 + z\left[\omega_{iN}S_{\omega_{iN}}(t)\alpha_i + C_{\omega_{iN}}(t)\alpha_N\right] + o(z^2)$$

and this result yields the known linear flow, consisting of 'rotations' around the origin of the iN plane,

$$\alpha_i' = C_{\omega_{iN}}(t)\alpha_i - S_{\omega_{iN}}(t)\alpha_N \qquad \alpha_N' = \omega_{iN}S_{\omega_{iN}}(t)\alpha_i + C_{\omega_{iN}}(t)\alpha_N \qquad \alpha_j' = \alpha_j.$$

4. Example: $U_z\big(\mathfrak{iso}_{\omega_2,\omega_3}(3)\big)$

In the previous section $U_z(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ has been studied; now we consider the particular case $N = 3$. The following discussion clarifies the concepts introduced up to now since the three-dimensional nature of the group T_{z3}. It allows us to represent graphically all the geometric constructions.

The Hopf algebra $U_z(\mathfrak{iso}_{\omega_2,\omega_3}(3))$ is generated by $P_1, P_2, P_3, J_{12}, J_{13}$ and J_{23}. The commutators and the rest of the structure tensors are obtained after setting the corresponding expressions of the previous section for $N = 3$. In this case, $U_z(\mathfrak{iso}_{\omega_2,\omega_3}(3)) = U_z(\mathfrak{so}_{\omega_2,\omega_3}(3)) \bowtie F(T_{z,3})$, where the group $T_{z,3}$ is characterized by the composition law

$$(\alpha_1', \alpha_2', \alpha_3')(\alpha_1, \alpha_2, \alpha_3) = \big(\alpha_1' + \mathrm{e}^{-z\alpha_3'}\alpha_1, \alpha_2' + \mathrm{e}^{-z\alpha_3'}\alpha_2, \alpha_3' + \alpha_3\big).$$

The translation generators constitute a system of global coordinates over $T_{z,3}$

$$P_1(\alpha_1,\alpha_2,\alpha_3) = \alpha_1 \qquad P_2(\alpha_1,\alpha_2,\alpha_3) = \alpha_2 \qquad P_3(\alpha_1,\alpha_2,\alpha_3) = \alpha_3.$$

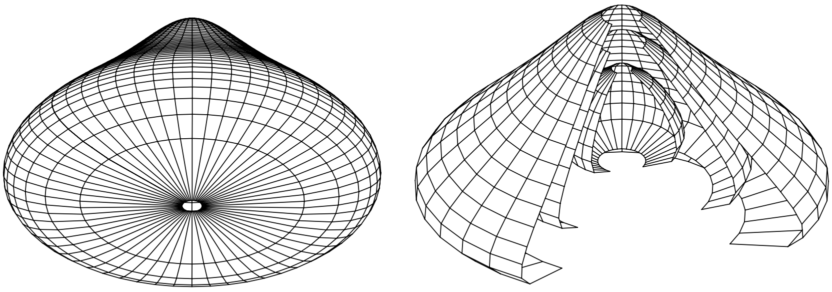

Figure 2. Typical leaf (left) and foliation (right) of $T_{z,3}$ for the case ($\omega_2 > 0, \omega_3 > 0; z > 0$).

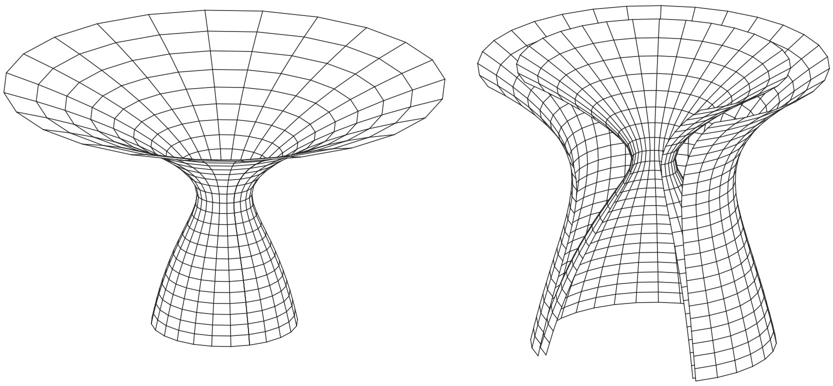

Figure 3. Leaf for ($\omega_2 > 0, \omega_3 < 0, z < 0$) (left). Foliation of $T_{z,3}$ (right).

With respect to these coordinates the action of $SO_{\omega_2,\omega_3}(3)$ on $T_{z,3}$, induced by the structure of the $U_z(\mathfrak{so}_{\omega_2,\omega_3}(3))$-algebra module of $F(T_{z,3})$, is given by the vector fields

$$\hat{J}_{12} = -P_2\frac{\partial}{\partial P_1} + \omega_{12}P_1\frac{\partial}{\partial P_2}$$
$$\hat{J}_{13} = -\left[\frac{1-\mathrm{e}^{-2zP_3}}{2z} + \frac{z}{2}\left(\omega_{13}P_1^2 - \omega_{23}P_2^2\right)\right]\frac{\partial}{\partial P_1} - z\omega_{13}P_1P_2\frac{\partial}{\partial P_2} + \omega_{13}P_1\frac{\partial}{\partial P_3}$$
$$\hat{J}_{23} = -z\omega_{23}P_2P_1\frac{\partial}{\partial P_1} - \left[\frac{1-\mathrm{e}^{-2zP_3}}{2z} + \frac{z}{2}\left(-\omega_{13}P_1^2 + \omega_{23}P_2^2\right)\right]\frac{\partial}{\partial P_2} + \omega_{23}P_2\frac{\partial}{\partial P_3}.$$

The (generalized) distribution generated by these fields is integrable since they close the algebra $\mathfrak{so}_{\omega_2,\omega_3}(3)$. The invariant

$$h_{\omega,z} = \omega_{13}P_1^2\,\mathrm{e}^{zP_3} + \omega_{23}P_2^2\,\mathrm{e}^{zP_3} + \left[\frac{\sinh\left(\frac{z}{2}P_3\right)}{\frac{z}{2}}\right]^2$$

allows us to analyse easily the nature of the leaves of the foliation. The two-dimensional leaves correspond to the connected components of the sets $h_{\omega,z}^{-1}(t) \subset T_{z,3}$, $t \in \mathbb{R}$ being a regular value of $h_{\omega,z}$. For example, when ($\omega_2 > 0, \omega_3 > 0; z > 0$) two strata appear: the origin point and the rest of the space. In figure 2 the foliation of $T_{z,3}$ is displayed for these cases. Figure 3 shows one of the typical leaves of the case ($\omega_2 > 0, \omega_3 < 0$) and part of the foliation. Figure 4 shows the two-dimensional leaves when $z = 0$, for each of the nine cases that appear by

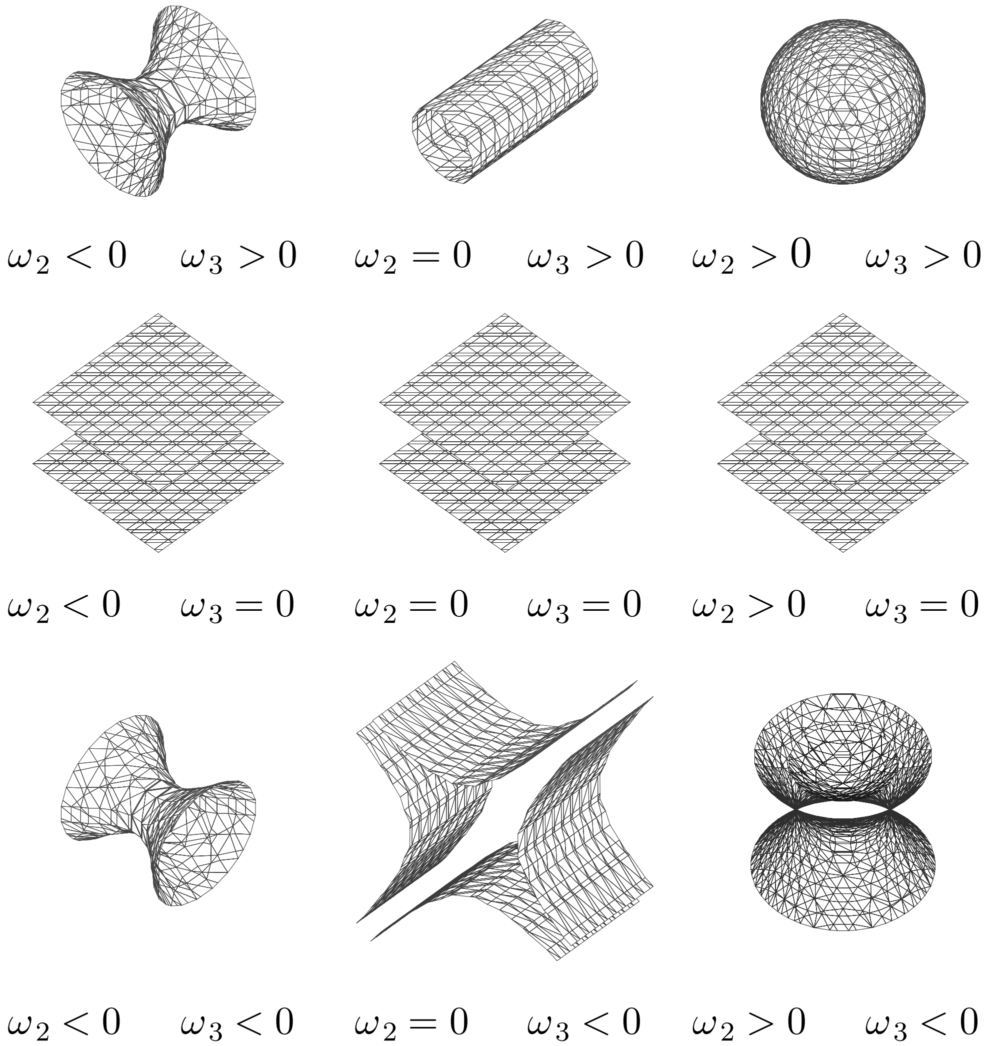

Figure 4. Typical leaves and foliation of $T_{z,3}$ in the nondeformed case ($z = 0$).

considering the different signs of ω_2 and ω_3. In the nondeformed case the study is reduced essentially to classifying the family of quadrics

$$\omega_{13}\alpha_1^2 + \omega_{23}\alpha_2^2 + \alpha_3^2 + c = 0.$$

When $c \neq 0$ every connected component constitutes a two-dimensional orbit of the action, but for $c = 0$ zero-dimensional orbits appear.

In figure 5 typical leaves of the deformed case $z > 0$ are displayed. As one can observe no qualitative differences are seen with respect to the nondeformed case. The leaves appearing in the cases with $z \neq 0$ are homotopic deformations of the corresponding cases with $z = 0$.

In these figures the symmetry exhibited by the leaves of each foliation can be observed: in the last column there is rotation symmetry with respect to the vertical axis, in the second column there is translation symmetry along one of the horizontal axes and in the first column there is symmetry with respect to hyperbolic rotations generated by the action of the generator J_{12}.

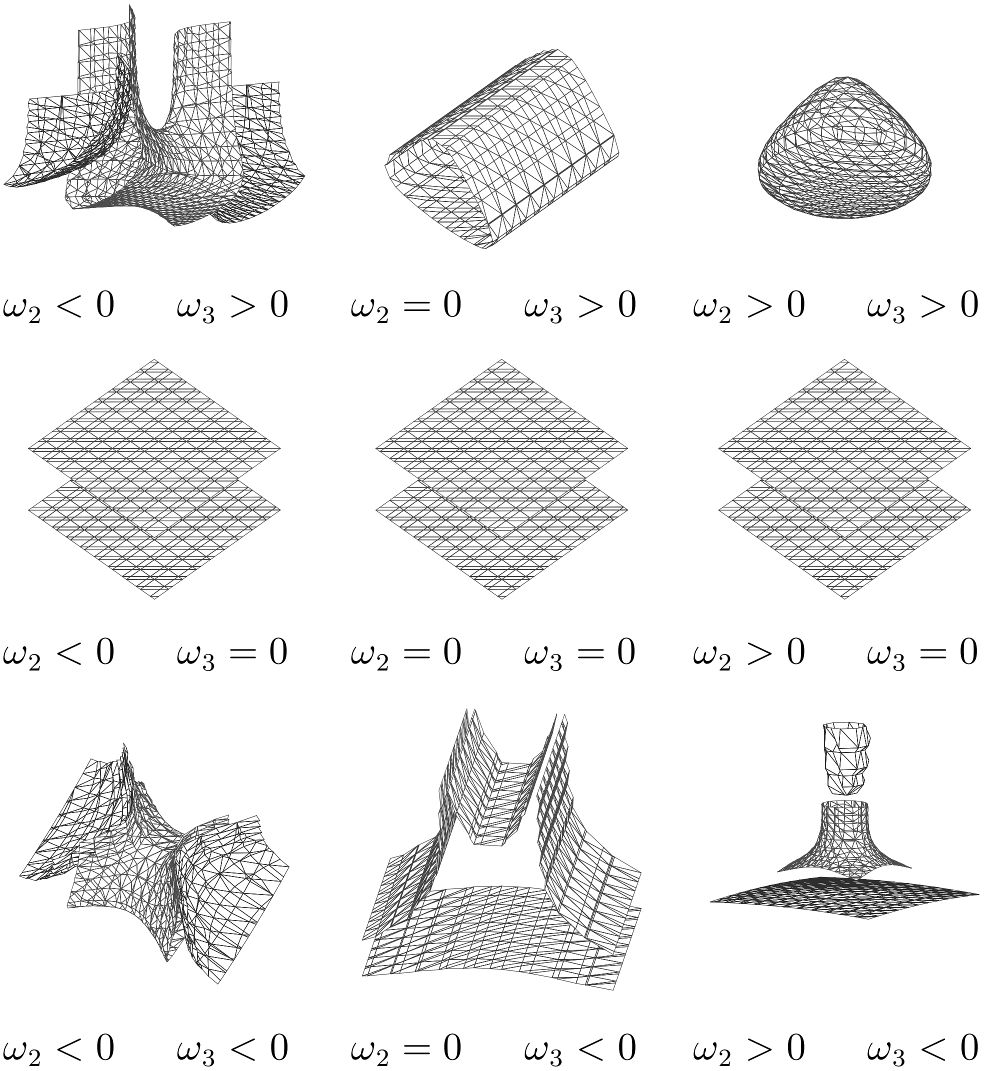

Figure 5. Typical leaves and foliation of $T_{z,3}$ for $z > 0$.

Expressions (3.2) and (3.15) allow us to describe the one-parameter flows associated with the generators. For $\hat{J}_{12}$ a linear action is obtained,

$$\Phi^t_{12}(\alpha_1, \alpha_2, \alpha_3) = \left(C_{\omega_{12}}(t)\alpha_1 - S_{\omega_{12}}(t)\alpha_2, \omega_{12} S_{\omega_{12}}(t)\alpha_1 + C_{\omega_{12}}(t)\alpha_2, \alpha_3\right)$$

unlike what happens for $\hat{J}_{13}$ and $\hat{J}_{23}$

$$\Phi^t_{13}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{-\left[\sinh(z\alpha_3) + \frac{z^2}{2}\left(\omega_{13}\alpha_1^2 + \omega_{23}\alpha_2^2\right) \mathrm{e}^{z\alpha_3}\right] S_{\omega_{13}}(t) + z\alpha_2 C_{\omega_{13}}(t)}{zF_{13}(\alpha, t)}, \right.$$
$$\left. \frac{\alpha_2\, \mathrm{e}^{z\alpha_3}}{F_{13}(\alpha, t)}, \frac{1}{z} \ln F_{13}(\alpha, t) \right)$$

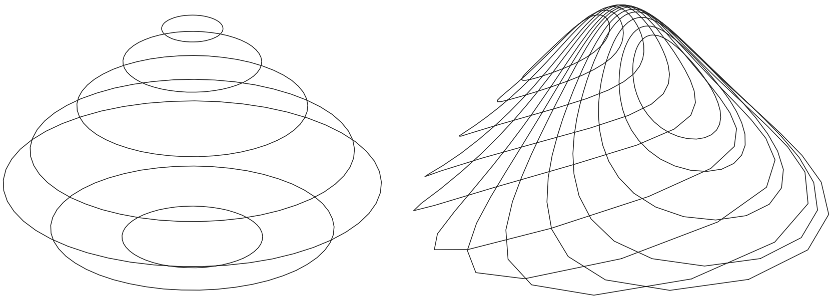

Figure 6. Foliations of the typical leaf for ($\omega_2 > 0, \omega_3 > 0; z > 0$) induced by Φ_{12} (left) and Φ_{13} (right).

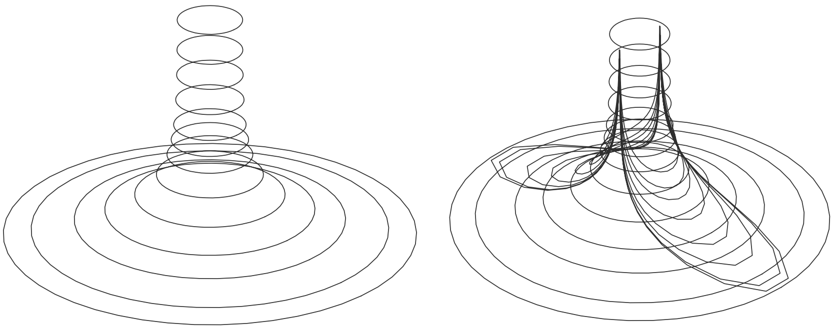

Figure 7. Foliations of a multiply connected leaf for ($\omega_2 > 0, \omega_3 < 0; z > 0$) induced by: Φ_{12} (left) and Φ_{13} (right together with the first one).

$$\Phi^t_{23}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{\alpha_1\, \mathrm{e}^{z\alpha_3}}{F_{23}(\alpha, t)}, \right.$$
$$\left. \frac{-\left[\sinh(z\alpha_3) + \frac{z^2}{2}\left(\omega_{13}\alpha_1^2 + \omega_{23}\alpha_2^2\right)\mathrm{e}^{z\alpha_3}\right] S_{\omega_{23}}(t) + z\alpha_2 C_{\omega_{23}}(t)}{zF_{23}(\alpha, t)}, \frac{1}{z}\ln F_{23}(\alpha, t) \right)$$

where

$$F_{i3}(\alpha, t) = \left[\cosh(z\alpha_3) + \frac{z^2}{2}\left(\omega_{13}\alpha_1^2 + \omega_{23}\alpha_2^2\right)\mathrm{e}^{z\alpha_3}\right]$$
$$+ \left[\sinh(z\alpha_3) - \frac{z^2}{2}\left(\omega_{13}\alpha_1^2 + \omega_{23}\alpha_2^2\right)\mathrm{e}^{z\alpha_3}\right] C_{\omega_{i3}}(t) + z\omega_{i3}\alpha_i\, \mathrm{e}^{z\alpha_3} S_{\omega_{i3}}(t).$$

Note that if $\omega_3 = 0$ then Φ^t_{13} and Φ^t_{23} are translations which generate the planes of the central row of figures 4 and 5.

The action of the one-parameter subgroups gives a new foliation of the two-dimensional leaves presented in the previous subsection. Figures 6–8 show the foliation of leaves of some of the algebras of the family considered.

The curves that appear in the foliation due to $\hat{J}_{13}$, for example, may be interpreted as the intersection of the surfaces determined by the invariants

$$\omega_{13}P_1^2\,\mathrm{e}^{zP_3} + \omega_{23}P_2^2\,\mathrm{e}^{zP_3} + \left[\frac{\sin\left(\frac{z}{2}P_3\right)}{\frac{z}{2}}\right]^2 \qquad P_2\,\mathrm{e}^{zP_3}.$$

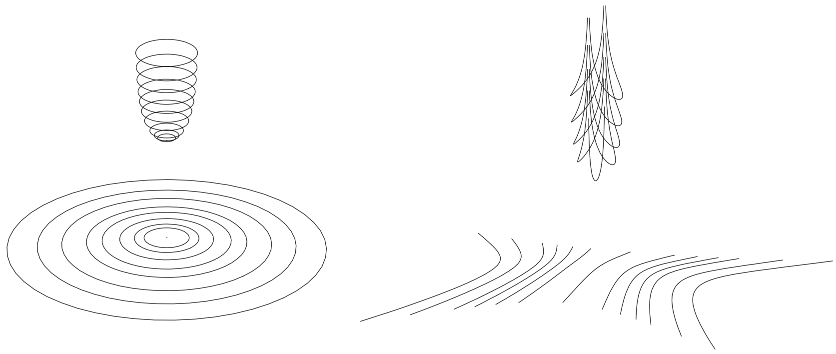

Figure 8. Foliations of simple connected leaves for ($\omega_2 > 0, \omega_3 < 0; z > 0$) induced by: Φ_{12} (left) and Φ_{13} (right).

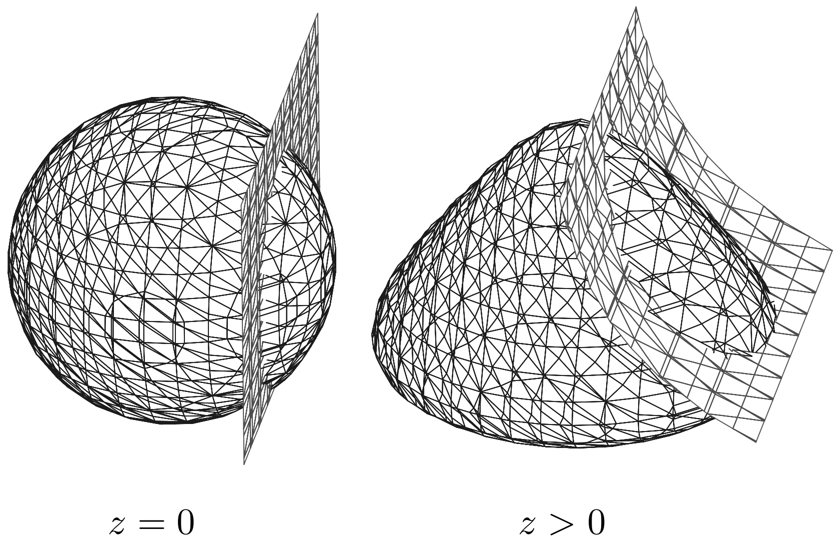

Figure 9. Intersection of the surfaces determined by the invariants for ($\omega_2 > 0, \omega_3 > 0$).

In figures 9–11 these intersections for different signs of ω_2 and ω_3 are displayed. In each figure, the nondeformed case as well as the deformed case are presented.

Note that the kinds of intersection are qualitatively the same in the deformed and nondeformed cases in figures 9 and 10. However, the last row of figure 11 does not have a nondeformed counterpart. These kinds of intersection are responsible for the appearance of the diagrams corresponding to the cases ($a < 0, b = 0$), ($a > 0, b = 0$), ($a < 0, b < 0$) and ($a > 0, b < 0$) in figure 1.

In summary, all the qualitative characteristics relative to the deformation with respect to the flow of the fields $\hat{J}_{ij}$, appear in the case $N = 3$.

5. Concluding remarks

It is worth noting that the reinterpretation of the bicrossproduct structure $\mathcal{H} = U(\mathcal{K}) \triangleright\!\!\blacktriangleleft U_z(\mathcal{L})$, in the case that $U_z(\mathcal{L})$ is a commutative (but noncocommutative) Hopf algebra, as

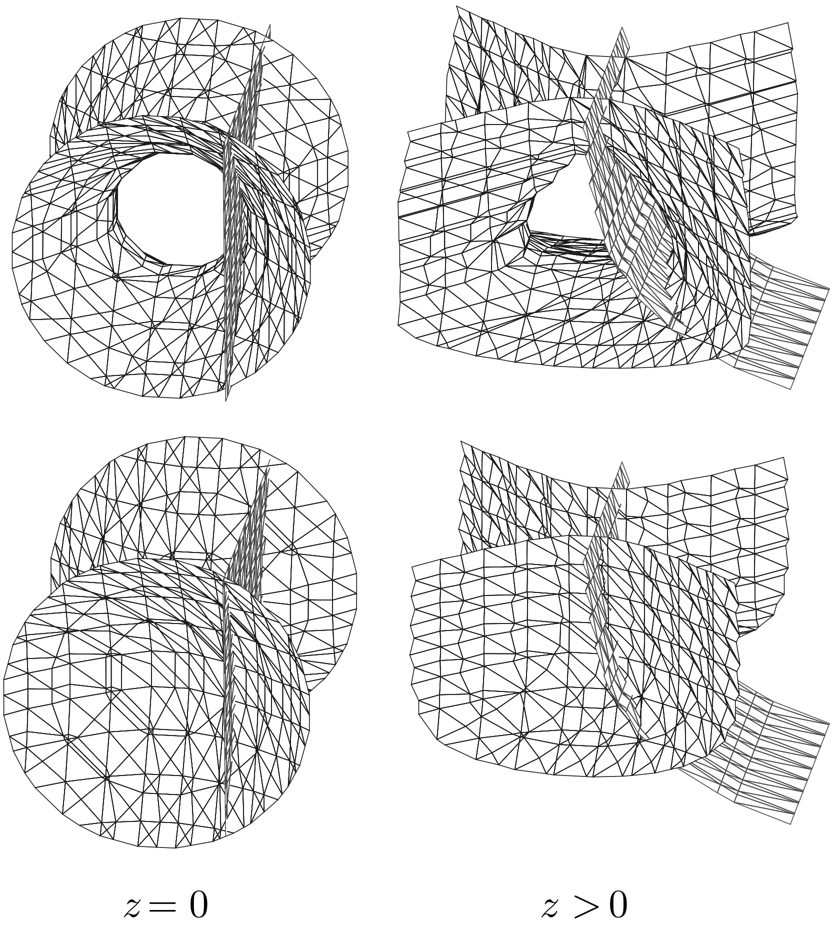

Figure 10. Intersection of the surfaces determined by the invariants for ($\omega_2 < 0, \omega_3 > 0$).

$H = U(\mathcal{K}) \bowtie \mathrm{Fun}(L_z)$ allows us to carry the action determining the bicrossproduct to an action of the group K on L_z.

For the algebras involved in this work, in the deformed case, i.e. $z \neq 0$, the above-mentioned action is local and nonlinear although in the opposite case the action is global and linear.

The flows have been obtained by studying the case of $\omega_i > 0$. An analytical dependence of the flow on the parameters ω_i is observed, which makes it unnecessary to repeat the computations for the other values of ω_i. This result is very interesting since the structure of the orbit space of the action of $SO_{\omega_2,\omega_3,\ldots,\omega_N}(N)$ on $T_{z,N}$ is very complicated, which makes it difficult to obtain directly the flows for each particular case.

In [5], the flows have been used for the computation of the induced representations for the $U_z(\mathfrak{iso}_\omega(2))$. For higher dimensions the problem of constructing the induced representations is very cumbersome and it is still an open problem.

The CK family $U_z(\mathfrak{iso}_{\omega_2,\omega_3}(3))$ contains, for instance, the q-Poincaré algebra ($\omega_2 < 0, \omega_3 > 0$), ($\omega_2 > 0, \omega_3 < 0$), ($\omega_2 < 0, \omega_3 < 0$), the q-Galilei algebra ($\omega_2 = 0, \omega_3 > 0$) and the q-Euclidean algebra ($\omega_2 > 0, \omega_3 > 0$). For a physical meaning of their generators see [11].

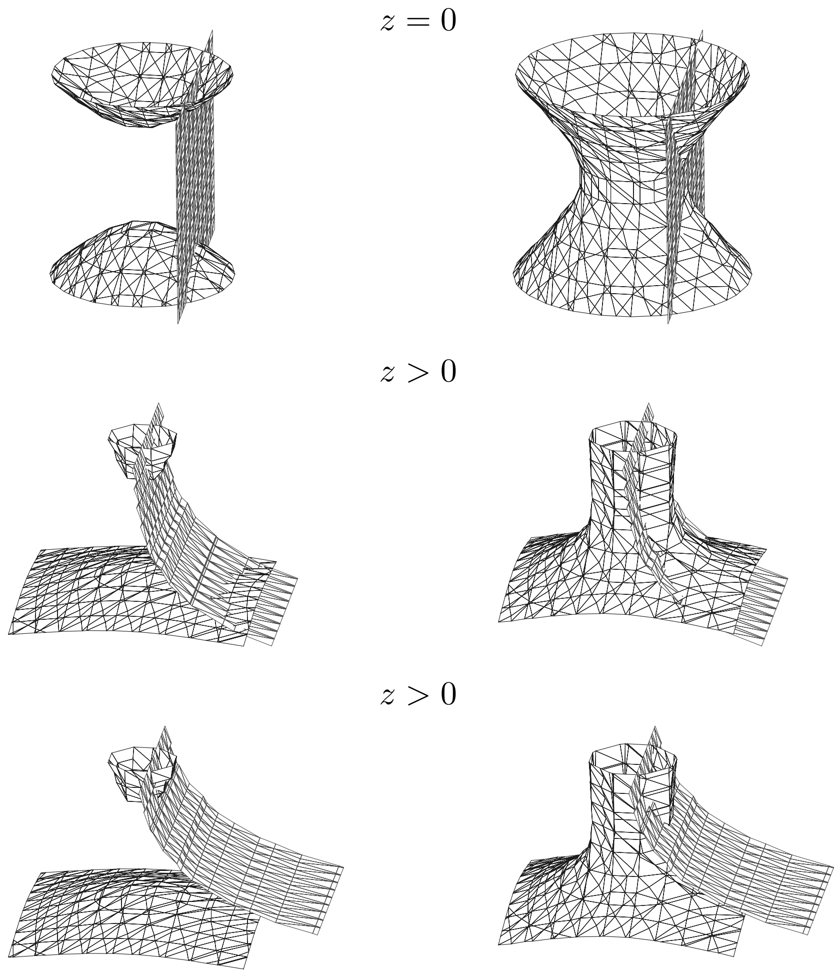

Figure 11. Intersection of the surfaces determined by the invariants for ($\omega_2 > 0, \omega_3 < 0$).

We finish with the following remarks about the system (3.10):

(1) The second-order systems associated with (3.10),

$$\ddot{x} = -ax\dot{x} - a(1 + 2x + ax^2)x \qquad \ddot{y} = -\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}a(1 + b\,\mathrm{e}^{-2y})$$

can be interpreted in both cases as moving objects over a straight line under the action of forces depending on the position and the velocity. In figure 12 the time evolution is represented for some significant cases.

(2) The system (3.10) is associated with the vector field over $\mathbb{R}^2$

$$X_{a,b} = \left[-\frac{1}{2}ax^2 - \frac{1}{2} + \frac{1}{2}b\,\mathrm{e}^{-2y}\right]\frac{\partial}{\partial x} + ax\frac{\partial}{\partial y}$$

which admits a Hamiltonian description as we are going to prove. Obviously, the pair (x, y) is not a chart of canonical coordinates since the 1-form obtained by the contraction of the vector field and the symplectic 2-form associated with this chart $(X_{a,b}\rfloor(\mathrm{d}x \wedge \mathrm{d}y))$ are not exact.

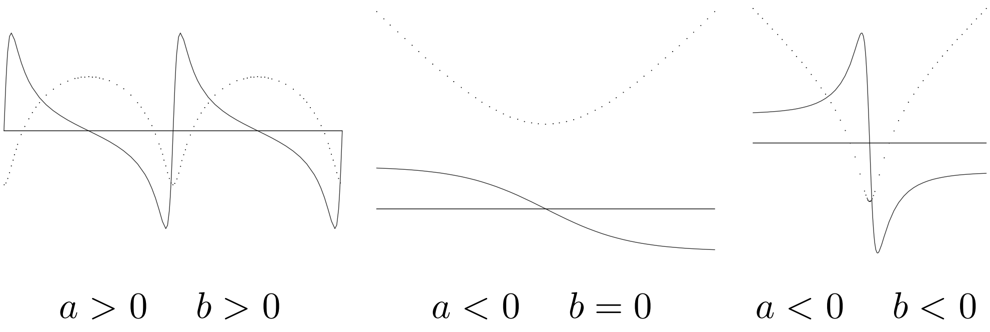

Figure 12. Time evolution of the systems $x(t)$ (continuous lines) and $y(t)$ (dotted lines).

Hence, let us consider a general symplectic 2-form $\omega = \Omega \, \mathrm{d}x \wedge \mathrm{d}y$, with Ω to be determined. Since $h_{a,b}$ is an invariant of the system it is evident that the Hamiltonian of the system has to be of the form $h = f \circ h_{a,b}$, with $f : \mathbb{R} \to \mathbb{R}$, which is not unequivocally determined. The vector field associated with h by means of the symplectic structure is fixed by $X_h \rfloor \omega = -\mathrm{d}h$. So,

$$X_h = -\frac{f' \circ h_{a,b}}{\Omega} \partial_y h_{a,b} \frac{\partial}{\partial x} + \frac{f' \circ h_{a,b}}{\Omega} \partial_x h_{a,b} \frac{\partial}{\partial y}.$$

Identifying X_h with $X_{a,b}$ two equations are obtained, but only one is independent. Hence,

$$\Omega = f' \circ h_{a,b} \frac{\partial_x h_{a,b}}{ax} = 2\,\mathrm{e}^y f' \circ h_{a,b}.$$

The simple choice $f(t) = t$ allows us to obtain the 2-form $\omega = 2\,\mathrm{e}^y \, \mathrm{d}x \wedge \mathrm{d}y$ that is independent of the parameters a and b. With the above selection of f the Hamiltonian of $X_{a,b}$ is the invariant $h_{a,b}$.

Acknowledgments

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